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Gravitational Energy-Momentum Densities of Arbitrary Weight

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Abstract

Various familiar expressions for the gravitational energy-momentum complex are derived from a generalised superpotential for arbitrary weight.

In order to obtain in Einstein's gravitation theory (Einstein, 1916) balance of energy-momentum for a system of matter in a gravitational field, a complete energy-momentum complex

$$_E\Theta_\mu^\nu = T_\mu^\nu + {}_Et_\mu^\nu \tag{1}$$

is introduced. The matter tensor T_μ^ν occurs in the field equations

$$G_\mu^\nu = -\kappa T_\mu^\nu \tag{2}$$

and the affine tensor $_Et_\mu^\nu$ of the gravitational field is (as a function of the metric tensor $g_{\alpha\beta}$) derived from a Lagrangian in such a way that the density $\sqrt{(-g)}_E\Theta_\mu^\nu$ of weight 1 satisfies the divergence relation

$$[\sqrt{(-g)}_E\Theta_\mu^\nu]_{,\nu} = 0 \tag{3}$$

If we eliminate T_μ^ν in (1) by means of the field equations (2), $_E\Theta_\mu^\nu$ appears as a function of the metric tensor and its derivatives only, and the conservation law (3) then holds identically. Therefore $\sqrt{(-g)}_E\Theta_\mu^\nu$ may be written as a divergence of an antisymmetric superpotential.

On the other hand we may start with a superpotential so that the divergence gives an affine tensor density which satisfies the divergence relation (3). This conserved quantity is conceived as the complete energy-momentum density and the contribution of the gravitational field is then obtained by subtracting the matter tensor.

In this paper we shall construct conserved affine tensor densities of arbitrary weight as a function of the curved metric tensor $g_{\alpha\beta}$ (and its derivatives). The expressions for ${}_E\Theta^\nu_\mu$ (Einstein, 1916), ${}_B\Theta^{\mu\nu}$ (Bergmann & Thomson, 1953), ${}_L\Theta^{\mu\nu}$ (Landau & Lifshitz, 1951) and ${}_M\Theta^\nu_\mu$ (Møller, 1958) are found to be special cases of the constructed quantities.

In general the affine tensor density of weight n

$$(\sqrt{-g})^n \Theta_n{}^{\mu\nu} = \Pi^{\mu\nu\sigma}{}_{,\sigma} \tag{4}$$

satisfies the conservation condition of weight n

$$[(\sqrt{-g})^n \Theta_n{}^{\mu\nu}]_{,\nu} = 0 \tag{5}$$

if the superpotential satisfies the antisymmetry condition

$$\Pi^{\mu\nu\sigma} = -\Pi^{\mu\sigma\nu} \tag{6}$$

The tensor density with mixed indices $(\sqrt{-g})^n g_{\mu\sigma}\Theta_n{}^{\sigma\nu}$ defined by ordinary index lowering is not conserved because $g_{\mu\nu,\sigma} \neq 0$.† Therefore

$$(\sqrt{-g})^n \Theta_n{}^\nu{}_\mu = (g_{\mu\tau}\Pi^{\tau\nu\sigma})_{,\sigma} = (\sqrt{-g})^n g_{\mu\tau}\Theta_n{}^{\tau\nu} + g_{\mu\tau,\sigma}\Pi^{\tau\nu\sigma} \tag{7}$$

will be considered as the appropriate mixed form of expression (4).

With the help of the tensor‡

$$g^{\mu\nu\sigma\tau} \equiv g^{\mu\nu}g^{\sigma\tau} - g^{\mu\sigma}g^{\nu\tau} \tag{8}$$

with symmetry properties

$$g^{\mu\nu\sigma\tau} = g^{\nu\mu\tau\sigma} = g^{\sigma\tau\mu\nu} = -g^{\mu\sigma\nu\tau} \text{ etc.} \tag{9}$$

we construct from the field variables $g_{\alpha\beta}$ and their first derivatives the following superpotentials, which satisfy the required anti-symmetry property (6) (because of (9)):

$$\Phi_n{}^{\mu\nu\sigma} = \tfrac{1}{2}(\sqrt{-g})^n g^{\mu\nu\sigma\tau}g^{\rho\alpha}g_{\tau\alpha,\rho} \tag{10}$$

$$X_n{}^{\mu\nu\sigma} = \tfrac{1}{2}(\sqrt{-g})^n g^{\rho\nu\sigma\tau}g^{\mu\alpha}g_{\tau\alpha,\rho} \tag{11}$$

$$\Psi_n{}^{\mu\nu\sigma} = \tfrac{1}{2}(\sqrt{-g})^n g^{\mu\nu\sigma\rho}g^{\tau\alpha}g_{\tau\alpha,\rho} = (\sqrt{-g})^n g^{\mu\nu\sigma\tau}\Gamma^\rho_{\tau\rho} \tag{12}$$

The general form of the superpotential $\Pi^{\mu\nu\sigma}$ is a linear combination of these affine tensor densities of weight n (10), (11) and (12)

$$(\sqrt{-g})^n \Theta_n{}^{\mu\nu} = (\alpha\Phi_n{}^{\mu\nu\sigma} + \beta X_n{}^{\mu\nu\sigma} + \gamma\Psi_n{}^{\mu\nu\sigma})_{,\sigma} \tag{13}$$

The gravitational part $t_n{}^{\mu\nu}$ of the complete energy-momentum complex $\Theta_n{}^{\mu\nu}$ is in analogy to (1) determined by

$$\Theta_n{}^{\mu\nu} = T^{\mu\nu} + t_n{}^{\mu\nu} \tag{14}$$

† Papapetrou (Papapetrou, 1948) gets round this difficulty by introducing a flat metric $\eta_{\mu\nu}$ (with $\eta_{\mu\nu,\sigma} = 0$).

‡ Notice that $R^{\mu\nu} = g^{\nu\sigma\tau\rho}(\Gamma^\mu_{\sigma\rho,\tau} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\rho\tau})$.

The indices of $T^{\mu\nu}$ are lowered and raised in the usual way. $t_n{}^{\mu\nu}$ contains only first-order derivatives of the metric tensor if the coefficients α, β, γ in (13) satisfy the condition

$$-\alpha = \beta = \gamma \tag{15}$$

For $n = 1$, equation (13) leads, with the choice (15) and $\alpha = -(1/\kappa)$, to the Bergmann expression ${}_B\Theta^{\mu\nu}$ (Bergmann & Thomson, 1953) and, according to (7), the Einstein expression ${}_E\Theta^{\nu}_{\mu}$ (Einstein, 1916) is just the mixed form of ${}_B\Theta^{\mu\nu}$. For $n = 1$, with

$$\alpha = \gamma = 0 \qquad \text{and} \qquad \beta = \frac{1}{\kappa} \tag{16}$$

(13) leads to the contravariant form of the Møller complex (Møller, 1958). None of these complexes are symmetric. The formulation of angular momentum in terms of a non-symmetric energy-momentum complex may involve difficulties. A complex $\Theta^{\mu\nu}_n$ in (13) is symmetric if

$$\begin{aligned} &\text{for } n \neq 0: \quad -\alpha = \beta = \frac{2}{n}\gamma \\ &\text{for } n = 0: \quad -\alpha = \beta; \qquad \gamma = 0 \end{aligned} \tag{17}$$

Now we choose $\alpha = -(1/\kappa)$ in agreement with the convention used in equation (2). Then (13) becomes, with (17),

$$(\sqrt{-g})^n 2\kappa\Theta_n{}^{\overline{\mu\nu}} = [(\sqrt{-g})^n g^{\mu\nu\sigma\tau}]_{,\tau\sigma} \tag{18}$$

where the bar denotes that the tensor is symmetric. From (18) follows

$$2\kappa\Theta_n{}^{\overline{\mu\nu}} = 2\kappa\Theta_0{}^{\overline{\mu\nu}} + nH^{\overline{\mu\nu}} \tag{19}$$

in which

$$2\kappa\Theta_0{}^{\overline{\mu\nu}} = g^{\mu\nu\sigma\tau}{}_{,\tau\sigma} \tag{20}$$

and

$$H^{\overline{\mu\nu}} = \overline{g^{\nu\sigma\tau\rho}\Gamma^{\mu}_{\tau\rho}\Gamma^{\alpha}_{\sigma\alpha}} + g^{\mu\nu\sigma\tau}\{(n-2)\Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\rho}_{\tau\rho} - 2\Gamma^{\alpha}_{\sigma\tau}\Gamma^{\rho}_{\alpha\rho} + n\Gamma^{\alpha}_{\sigma\alpha,\tau}\} \tag{21}$$

where the bar over the whole expression with free indices μ and ν denotes the symmetric combination

$$\overline{A^{\mu\nu}} = A^{\mu\nu} + A^{\nu\mu} \tag{22}$$

In a coordinate system with $\sqrt{-g} = 1$, as initially chosen by Einstein (Einstein, 1916), one has $\Gamma^{\alpha}_{\sigma\alpha} = 0$. Then $H^{\overline{\mu\nu}} = 0$ and the difference between tensors and their densities disappears as it should do.

In order to find the gravitational term $t^{\overline{\mu\nu}}$ we rewrite the field equations (2), by using equation (14)

$$2G^{\overline{\mu\nu}} = -2\kappa\Theta_n{}^{\overline{\mu\nu}} + 2\kappa t_n{}^{\overline{\mu\nu}} \tag{23}$$

Some algebra shows that the left-hand side of (23) can be written as

$$2G^{\overline{\mu\nu}} = -g^{\mu\nu\sigma\tau}{}_{,\sigma\tau} + \overline{g^{\nu\sigma\tau\rho}(\Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\tau\rho} + \Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\tau\alpha})} + g^{\alpha\sigma\rho\tau}\Gamma^{\mu}_{\alpha\rho}\Gamma^{\nu}_{\sigma\tau}$$
$$+ g^{\mu\nu\sigma\tau}(\Gamma^{\alpha}_{\sigma\rho}\Gamma^{\rho}_{\alpha\tau} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\rho}_{\tau\rho} + 2\Gamma^{\alpha}_{\sigma\tau}\Gamma^{\rho}_{\alpha\rho} - 2\Gamma^{\alpha}_{\sigma\alpha,\tau}) \qquad (24)$$

From (20), (23) and (24) it immediately follows that

$$2\kappa t_0{}^{\overline{\mu\nu}} = \overline{g^{\nu\sigma\tau\rho}(\Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\tau\rho} + \Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\tau\alpha})} + g^{\alpha\sigma\rho\tau}\Gamma^{\mu}_{\alpha\rho}\Gamma^{\nu}_{\sigma\tau}$$
$$+ g^{\mu\nu\sigma\tau}(\Gamma^{\alpha}_{\sigma\rho}\Gamma^{\rho}_{\alpha\tau} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\rho}_{\tau\rho} + 2\Gamma^{\alpha}_{\sigma\tau}\Gamma^{\rho}_{\alpha\rho} - 2\Gamma^{\alpha}_{\sigma\alpha,\tau}) \quad (25)$$

In analogy with equation (19) the gravitational $t_n{}^{\overline{\mu\nu}}$, because of (14), can be written as

$$2\kappa t_n{}^{\overline{\mu\nu}} = 2\kappa t_0{}^{\overline{\mu\nu}} + nH^{\overline{\mu\nu}} \qquad (26)$$

Now the general expression for the symmetric affine tensor of the gravitational field is determined by equations (26), (25) and (21):

$$2\kappa t_n{}^{\overline{\mu\nu}} \doteq \overline{g^{\nu\sigma\tau\rho}[\Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\tau\rho} + (1-n)\Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\tau\alpha}]} + g^{\alpha\sigma\rho\tau}\Gamma^{\mu}_{\alpha\rho}\Gamma^{\nu}_{\sigma\tau}$$
$$+ g^{\mu\nu\sigma\tau}[\Gamma^{\alpha}_{\sigma\rho}\Gamma^{\rho}_{\alpha\tau} + 2(1-n)\Gamma^{\alpha}_{\sigma\tau}\Gamma^{\rho}_{\alpha\rho} + (1-n)^2\Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\rho}_{\tau\rho}$$
$$+ (n-2)\Gamma^{\alpha}_{\sigma\alpha,\tau}] \qquad (27)$$

Landau and Lifshitz (Landau & Lifshitz, 1951) define the gravitational term as the difference between the expressions for $G^{\overline{\mu\nu}}/\kappa$ in a local inertial system ($g_{\mu\nu,\sigma} = 0$) and an arbitrary system of reference respectively. As a consequence ${}_Lt^{\overline{\mu\nu}}$ is symmetric and disappears in a local inertial system. Therefore, ${}_Lt^{\overline{\mu\nu}}$ contains only first-order derivatives of the metric tensor. It follows from (27) that $t_2^{\overline{\mu\nu}}$ is the only symmetric expression, which contains no second-order derivatives of $g_{\alpha\beta}$. So it is obvious that ${}_L\Theta^{\overline{\mu\nu}}$ ($= \Theta_2{}^{\overline{\mu\nu}}$) leads to a conservation condition of weight 2. The physical significance of a density of weight $n \neq 1$ is not quite clear. Furthermore, these densities have physical significance only when integrated over the spatial coordinates with appropriate asymptotic conditions. These integrated quantities can be considered as the energy-momentum of the total system under the usual requirements.

In case a flat reference system is introduced (for example with respect to flat asymptotic conditions), the Minkowski metric tensor $\eta^{\mu\nu}$ may be used in addition to the curved $g^{\mu\nu}$ for the construction of a still wider class of super-potentials. All factors $g^{\kappa\lambda}$ (but of course not their derivatives) in (8) and (10), (11) and (12) might be replaced by the corresponding $\eta^{\kappa\lambda}$. Condition (17) then reduces to $-\alpha = \beta$. With the choice (15) and $n = 0$ this leads to Weinberg's energy-momentum complex ${}_W\Theta^{\mu\nu}$ (Weinberg, 1972).

Further investigation is needed to see which of the proposed expressions for the complete energy-momentum density of a certain weight (symmetrical or not) has preference, for example on the grounds of the criteria mentioned in Møller (Møller, 1966).

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